

MOMENT-ANGLE MANIFOLDS AND PANOV'S PROBLEM

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ABSTRACT. We answer a problem posed by Panov, which is to describe the relationship between the wedge summands in a homotopy decomposition of the moment-angle complex corresponding to a disjoint union of ℓ points and the connected sum factors in a diffeomorphism decomposition of the moment-angle manifold corresponding to the simple polytope obtained by making ℓ vertex cuts on a standard d -simplex. This establishes a bridge between two very different approaches to moment-angle manifolds.

1. INTRODUCTION

Moment-angle complexes have attracted a great deal of interest recently because they are a nexus for important problems arising in algebraic topology, algebraic geometry, combinatorics, complex geometry and commutative algebra. They are best described as a special case of the polyhedral product functor, popularized in [BBCG] as a generalization of moment-angle complexes and K -powers [BP2], which were in turn generalizations of moment-angle manifolds [DJ].

Let K be a simplicial complex on m vertices. For $1 \leq i \leq m$, let (X_i, A_i) be a pair of pointed CW -complexes, where A_i is a pointed subspace of X_i . Let $(\underline{X}, \underline{A}) = \{(X_i, A_i)\}_{i=1}^m$ be the sequence of CW -pairs. For each simplex (face) $\sigma \in K$, let $(\underline{X}, \underline{A})^\sigma$ be the subspace of $\prod_{i=1}^m X_i$ defined by

$$(\underline{X}, \underline{A})^\sigma = \prod_{i=1}^m Y_i \quad \text{where} \quad Y_i = \begin{cases} X_i & \text{if } i \in \sigma \\ A_i & \text{if } i \notin \sigma. \end{cases}$$

The *polyhedral product* determined by $(\underline{X}, \underline{A})$ and K is

$$(\underline{X}, \underline{A})^K = \bigcup_{\sigma \in K} (\underline{X}, \underline{A})^\sigma \subseteq \prod_{i=1}^m X_i.$$

For example, suppose each A_i is a point. If K is a disjoint union of n points then $(\underline{X}, \underline{*})^K$ is the wedge $X_1 \vee \cdots \vee X_n$, and if K is the standard $(n-1)$ -simplex then $(\underline{X}, \underline{*})^K$ is the product $X_1 \times \cdots \times X_n$.

In the case when each pair of spaces (X_i, A_i) equals (D^2, S^1) , the polyhedral product $(\underline{X}, \underline{A})^K$ is called a *moment-angle complex*, and is written in more traditional notation as \mathcal{Z}_K . Two important properties of \mathcal{Z}_K are: the cohomology ring $H^*(\mathcal{Z}_K; \mathbb{Z})$ is the Tor-algebra $\text{Tor}_{\mathbb{Z}[v_1, \dots, v_m]}(\mathbb{Z}[K], \mathbb{Z})$ where $\mathbb{Z}[K]$ is the Stanley-Reisner face ring of K and $|v_i| = 2$ for $1 \leq i \leq m$; and \mathcal{Z}_K is homotopy equivalent to the complement of the coordinate subspace arrangement in \mathbb{C}^m determined by K .

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Stanley-Reisner face rings are a subject of intense interest in commutative algebra (even having its own MSC number), and complements of coordinate subspace arrangements are an area of major importance in combinatorics. The connection to moment-angle complexes allows for topological methods to be used to inform upon problems posed in commutative algebra and combinatorics.

To date, a great deal of work has been done to determine when \mathcal{Z}_K is homotopy equivalent to a wedge of spheres, or to produce analogous statements in the case of certain polyhedral products [GT2, GT3, GPTW, GW, IK1, IK2]. When this is the case, the complement of the corresponding coordinate subspace arrangement is homotopy equivalent to a wedge of spheres, and $H^*(\mathcal{Z}_K; \mathbb{Z})$ is *Golod*, meaning that all cup products and higher Massey products are zero. In all cases thus far, the arguments start by using combinatorics to identify a good class of simplicial complexes to consider, then homotopy theory is used to prove that \mathcal{Z}_K is homotopy equivalent to a wedge of spheres for this class of simplicial complexes, and finally it is deduced that $H^*(\mathcal{Z}_K; \mathbb{Z})$ is Golod.

Moment-angle complexes arise in complex geometry and algebraic geometry in a different way. Let P be a simple polytope, let P^* be its dual, and let ∂P^* be the boundary complex of P^* . Then $K = \partial P^*$ is a simplicial complex, and we let $\mathcal{Z}(P) = \mathcal{Z}_{\partial P^*}$. In this case, $\mathcal{Z}(P)$ has the richer structure of a manifold, and is called a *moment-angle manifold*. These manifolds can be interpreted as intersections of complex quadrics, each fibring over a projective toric variety. The topology and geometry of these manifolds have been studied in considerable depth [BM, BP2, DJ, GL]. In particular, in [BM, GL] a large class of simple polytopes P was identified for which $\mathcal{Z}(P)$ is diffeomorphic to a connected sum of products of two spheres.

Panov [P] observed that the two directions of work produce very similar results in the following way. If K is a simplicial complex consisting of ℓ disjoint points, then by [GT1] there is a homotopy equivalence

$$\mathcal{Z}_K \simeq \bigvee_{k=3}^{\ell+1} (S^k)^{\wedge(k-2)} \binom{\ell}{k-1}$$

where $(S^k)^{\wedge n}$ is the n -fold smash product of S^k with itself. On the other hand, if P is a simple polytope that has been obtained from the d -simplex by iteratively cutting off a vertex $\ell - 1$ times (the cuts occurring in any order), then by [BM] there is a diffeomorphism

$$\mathcal{Z}(P) \cong \#_{k=3}^{\ell+1} (S^k \times S^{\ell+2d-k})^{\#(k-2)} \binom{\ell}{k-1}$$

where $(S^k \times S^{\ell+2d-k})^{\#n}$ is the n -fold connected sum of $S^k \times S^{\ell+2d-k}$ with itself. The coefficients and the sphere dimensions in both decompositions coincide. This led Panov to pose the following.

Problem: Describe the nature of this correspondence.

The purpose of the paper is to answer this problem, thereby establishing a bridge between two very different approaches to moment-angle manifolds. Let P be a simple polytope obtained from a d -simplex by $\ell - 1$ vertex cuts. To study polyhedral products, we consider the dual simplicial

complex P^* , which is a *stacked polytope* (defined explicitly in Section 3). We show that the homotopy type of $\mathcal{Z}_{\partial P^*}$ is independent of the stacking order for P^* (dual to the result in [BM, GL] that the diffeomorphism type of $\mathcal{Z}(P)$ is independent of the order in which the vertex cuts occur for P). This lets us choose a stacking order, yielding a stacked polytope \mathcal{L} on the vertex set $[m]$ for $m = d + \ell$, which is more convenient to analyze (see Section 4 for details). We prove the following. Let $\partial\mathcal{L} - \{1\}$ be the full subcomplex of $\partial\mathcal{L}$ obtained by deleting the vertex $\{1\}$.

Theorem 1.1. *The stacked polytope \mathcal{L} has the following properties:*

- (a) *there is a homotopy equivalence $\mathcal{Z}_{\partial\mathcal{L} - \{1\}} \simeq \mathcal{Z}_{P_\ell}$ where P_ℓ is ℓ disjoint points;*
- (b) *the inclusion $\partial\mathcal{L} - \{1\} \rightarrow \partial\mathcal{L}$ induces a map $\mathcal{Z}_{\partial\mathcal{L} - \{1\}} \rightarrow \mathcal{Z}_{\partial\mathcal{L}}$, which up to homotopy equivalences, is a map*

$$f: \bigvee_{k=3}^{\ell+1} (S^k)^{\wedge(k-2)} \binom{\ell}{k-1} \rightarrow \#_{k=3}^{\ell+1} (S^k \times S^{\ell+2d-k})^{\#(k-2)} \binom{\ell}{k-1};$$

- (c) *f has a left homotopy inverse g ;*
- (d) *when restricted to a factor $H^*(S^k \times S^{\ell+2d-k})$ in the cohomology of the connected sum, f^* is zero on precisely one of the ring generators.*

It is helpful to point out one consequence of Theorem 1.1. Since f has a left homotopy inverse, f^* is an epimorphism. By part (d), f^* is nonzero on precisely one ring generator when restricted to any factor $H^*(S^k \times S^{\ell+2d-k})$. Let A be the collection of such generators, one from each factor in the connected sum. The matching coefficients in the wedge decomposition of $\mathbb{Z}_{\partial\mathcal{L} - \{1\}}$ and the connected sum decomposition of $\mathbb{Z}_{\partial\mathcal{L}}$ then implies that f^* maps A isomorphically onto $H^*(\bigvee_{k=3}^{\ell+1} (S^k)^{\wedge(k-2)} \binom{\ell}{k-1})$.

Along the way, we phrase as many of the intermediate results as possible in terms of polyhedral products $(\underline{CX}, \underline{X})^K$, where CX is the cone on a space X , or in terms of $(CX, X)^K$, where all the coordinate spaces X_i equal a common space X . This is of interest because, when $K = \partial P^*$ for P a simple polytope obtained from a d -simplex by $\ell - 1$ vertex cuts, $(CX, X)^K$ is analogous to the connected sum of products of two spheres. This analogue is *not* a connected sum, in general, nor is it even a manifold. So understanding its homotopy theory helps distinguish how much of the homotopy theory of a connected sum depends on the actual geometry.

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2. PRELIMINARY HOMOTOPY THEORY

In this section we give preliminary results regarding the homotopy theory of polyhedral products that will be used later on. In particular, in Proposition 2.4 we identify a family of simplicial complexes whose polyhedral products have the same homotopy type as the polyhedral product corresponding to a disjoint union of points.

For spaces A and B , the *right half-smash* $A \rtimes B$ is the space $(A \times B)/\sim$ where $(*, b) \sim *$. The *join* $A * B$ is the space $(A \times I \times B)/\sim$ where $(a, 1, b) \sim (*, 1, b)$ and $(a, 0, b) \sim (a, 0, *)$; it is well known that there is a homotopy equivalence $A * B \simeq \Sigma A \wedge B$. The following lemma was proved in [GT2].

Lemma 2.1. *Suppose that there is a homotopy pushout*

$$\begin{array}{ccc} A \times B & \xrightarrow{\pi_1} & A \\ \downarrow * \times 1 & & \downarrow \\ C \times B & \longrightarrow & Q \end{array}$$

where π_1 is the projection onto the first factor. Then there is a homotopy equivalence $Q \simeq (A * B) \vee (C \rtimes B)$. \square

Suppose that K is a simplicial complex on the vertex set $\{1, \dots, m\}$. If L is a sub-complex of K on vertices $\{i_1, \dots, i_k\}$ then when applying the polyhedral product to K and L simultaneously, we must regard L as a simplicial complex \bar{L} on the vertices $\{1, \dots, m\}$. By definition of the polyhedral product, we therefore obtain

$$(\underline{CX}, \underline{X})^{\bar{L}} = (\underline{CX}, \underline{X})^L \times \prod_{t=1}^{m-k} X_{j_t}$$

where $\{j_1, \dots, j_{m-k}\}$ is the complement of $\{i_1, \dots, i_k\}$ in $\{1, \dots, m\}$.

The following lemma describes the homotopy type of $(\underline{CX}, \underline{X})^K$ when $K = K_1 \cup \Delta^k$, where K_1 and Δ^k have been glued along a common face Δ^{k-1} . A similar gluing lemma was proved in [GT2] that was stated more generally in terms of two simplicial complexes joined along a common face, although it was stated only in the more restrictive case of $(\underline{C\Omega X}, \underline{\Omega X})$. For our purposes, it is helpful to be more explicit about the vertices in Δ^{k-1} , which affects the homotopy type of $(\underline{CX}, \underline{X})^K$, so a proof is included.

Lemma 2.2. *Let K be a simplicial complex on the vertex set $\{1, \dots, m\}$. Suppose that $K = K_1 \cup \Delta^k$ where: (i) K_1 is a simplicial complex on the vertex set $\{1, \dots, m-1\}$ and $\{i\} \in K_1$ for $1 \leq i \leq m-1$; (ii) Δ^k is on the vertex set $\{m-k, \dots, m\}$, and (iii) $K_1 \cap \Delta^k$ is a $(k-1)$ -simplex on the vertex set $\{m-k, \dots, m-1\}$. Then there is a homotopy equivalence*

$$(\underline{CX}, \underline{X})^K \simeq \left(\left(\prod_{i=1}^{m-k-1} X_i \right) * X_m \right) \vee \left((\underline{CX}, \underline{X})^{K_1} \rtimes X_m \right).$$

Proof. The simplicial complex K can be written as a pushout

$$\begin{array}{ccc} \Delta^{k-1} & \longrightarrow & \Delta^k \\ \downarrow & & \downarrow \\ K_1 & \longrightarrow & K. \end{array}$$

Regarding K_1 , Δ^k and Δ^{k-1} as simplicial complexes on the vertex set $\{1, \dots, m\}$ and applying the polyhedral product functor, we obtain a pushout

$$(1) \quad \begin{array}{ccc} (\underline{CX}, \underline{X})^{\overline{\Delta^{k-1}}} & \longrightarrow & (\underline{CX}, \underline{X})^{\overline{\Delta^k}} \\ \downarrow & & \downarrow \\ (\underline{CX}, \underline{X})^{\overline{K_1}} & \longrightarrow & (\underline{CX}, \underline{X})^{\overline{K}}. \end{array}$$

We now identify the spaces and maps in (1).

By hypothesis, K_1 is a simplicial complex on the vertex set $\{1, \dots, m-1\}$, Δ^k is on the vertex set $\{m-k, \dots, m\}$ and Δ^{k-1} is on the vertex set $\{m-k, \dots, m-1\}$. So by definition of the polyhedral product we have

$$\begin{aligned} (\underline{CX}, \underline{X})^{\overline{\Delta^{k-1}}} &= \prod_{i=1}^{m-k-1} X_i \times \prod_{i=m-k}^{m-1} CX_i \times X_m \\ (\underline{CX}, \underline{X})^{\overline{\Delta^k}} &= \prod_{i=1}^{m-k-1} X_i \times \prod_{i=m-k}^m CX_i \\ (\underline{CX}, \underline{X})^{\overline{K_1}} &= (\underline{CX}, \underline{X})^{K_1} \times X_m. \end{aligned}$$

Further, under these identifications, the map $(\underline{CX}, \underline{X})^{\overline{\Delta^{k-1}}} \rightarrow (\underline{CX}, \underline{X})^{\overline{\Delta^k}}$ is the identity on each factor indexed by $1 \leq i \leq m-1$ and is the inclusion $X_m \rightarrow CX_m$ on the m^{th} factor, and the map $(\underline{CX}, \underline{X})^{\overline{\Delta^{k-1}}} \rightarrow (\underline{CX}, \underline{X})^{\overline{K_1}}$ is the identity on the m^{th} factor. Therefore, as the cone CX_i is contractible, up to homotopy equivalences (1) is the same as the homotopy pushout

$$(2) \quad \begin{array}{ccc} (\prod_{i=1}^{m-k-1} X_i) \times X_m & \xrightarrow{\pi_1} & \prod_{i=1}^{m-k-1} X_i \\ \downarrow f \times 1 & & \downarrow \\ (\underline{CX}, \underline{X})^{K_1} \times X_m & \longrightarrow & (\underline{CX}, \underline{X})^K \end{array}$$

where π_1 is the projection and f is some map.

By [GT3], any simplicial complex L on vertices $\{1, \dots, \ell\}$ for which $\{i\} \in L$ for $1 \leq i \leq \ell$ has the property that the inclusion $\prod_{i=1}^\ell X_i \rightarrow (\underline{CX}, \underline{X})^L$ is null homotopic. In our case, by hypothesis, $\{i\} \in K_1$ for $1 \leq i \leq m-1$, so the inclusion $\prod_{i=1}^{m-1} X_i \rightarrow (\underline{CX}, \underline{X})^{K_1}$ is null homotopic. Since the map f in (2) factors through this inclusion, it too is null homotopic. Therefore Lemma 2.1 applies to the homotopy pushout (2), giving a homotopy equivalence

$$(\underline{CX}, \underline{X})^K \simeq \left(\left(\prod_{i=1}^{m-k-1} X_i \right) * X_m \right) \vee \left((\underline{CX}, \underline{X})^{K_1} \rtimes X_m \right).$$

□

For example, let P_m be m disjoint points. Then $P_m = P_{m-1} \cup \Delta^0$ where Δ^0 is a single point, and the union is taken over the emptyset. Applying Lemma 2.2 then immediately gives the following.

Corollary 2.3. *There is a homotopy equivalence*

$$(\underline{CX}, \underline{X})^{P_m} \simeq \left(\left(\prod_{i=1}^{m-1} X_i \right) * X_m \right) \vee \left((\underline{CX}, \underline{X})^{P_{m-1}} \rtimes X_m \right).$$

□

In Proposition 2.4 we will consider the polyhedral product $(\underline{CX}, \underline{X})^K$ where all the coordinate spaces X_i are equal to a common space X . In this case, we write $(CX, X)^K$. In particular, in the case of m disjoint points, Corollary 2.3 implies that there is a homotopy equivalence

$$(3) \quad (CX, X)^{P_m} \simeq \left(\left(\prod_{i=1}^{m-1} X \right) * X \right) \vee \left((CX, X)^{P_{m-1}} \rtimes X \right).$$

Proposition 2.4. *Let $k \geq 1$ and suppose that there is a sequence of simplicial complexes*

$$K_1 = \Delta^k \subseteq K_2 \subseteq \cdots \subseteq K_\ell$$

such that, for $i > 1$, $K_i = K_{i-1} \cup_{\sigma_i} \Delta^k$ where $\sigma_i = \Delta^{k-1}$. That is, K_i is obtained from K_{i-1} by gluing on a Δ^k along the common face σ_i . Let $K = K_\ell$ and observe that K is a simplicial complex on $k + \ell$ vertices. Then there is a homotopy equivalence

$$(CX, X)^K \simeq (CX, X)^{P_\ell}.$$

Remark 2.5. It may be useful to note that Proposition 2.4 also makes sense for $k = 0$, in which case Δ^0 is a point and each σ_i is the emptyset, in which case $K = K_\ell$ is ℓ disjoint points, and the conclusion is a tautology. In the case when $k = 1$, notice that $K = K_\ell$ is formed by iteratively taking an interval at stage i and gluing one of its endpoints to a vertex of the preceeding simplicial complex at stage $i - 1$. One example of this is the boundary of the $(\ell + 2)$ -gon with one vertex removed, another is all ℓ intervals joined at a common vertex.

Proof. Fix $k \geq 1$. The proof is by induction on ℓ . When $\ell = 1$, we have $K = K_1 = \Delta^k$. By definition of the polyhedral product, $(\underline{CX}, \underline{X})^K = \prod_{i=1}^{k+1} CX_i$, so $(CX, X)^K = \prod_{i=1}^{k+1} CX$. On the other hand, as P_1 is a single point, $(\underline{CX}, \underline{X})^{P_1} = (CX, X)^{P_1} = CX$. Thus $(CX, X)^K \simeq (CX, X)^{P_1}$ as both spaces are contractible.

Suppose that the proposition holds for all integers t satisfying $t < \ell$. Consider $K_\ell = K_{\ell-1} \cup_{\sigma_\ell} \Delta^k$ where $\sigma_\ell = \Delta^{k-1}$. Reordering the vertices if necessary, we may assume that $K_{\ell-1}$ is a simplicial complex on the vertex set $\{1, \dots, k + \ell - 1\}$, Δ^k is on the vertex set $\{\ell, \dots, k + \ell\}$, and $\sigma_\ell = \Delta^{k-1}$ is on the vertex set $\{\ell, \dots, k + \ell - 1\}$. By Lemma 2.2, there is a homotopy equivalence

$$(\underline{CX}, \underline{X})^{K_\ell} \simeq \left(\left(\prod_{i=1}^{\ell-1} X_i \right) * X_{k+\ell} \right) \vee \left((\underline{CX}, \underline{X})^{K_{\ell-1}} \rtimes X_{k+\ell} \right).$$

Therefore, there is a homotopy equivalence

$$(CX, X)^{K_\ell} \simeq \left(\left(\prod_{i=1}^{\ell-1} X \right) * X \right) \vee \left((CX, X)^{K_{\ell-1}} \rtimes X \right).$$

This formula is exactly the same as that in (3) for $(CX, X)^{P_\ell}$. By inductive hypothesis, $(CX, X)^{K_{\ell-1}} \simeq (CX, X)^{P_{\ell-1}}$, so we obtain $(CX, X)^{K_\ell} \simeq (CX, X)^{P_\ell}$. The proposition therefore holds by induction. \square

3. VERTEX CUTS AND STACKED POLYTOPES

In this section we discuss some constructions obtained from simple polytopes, and discuss some of their properties in the context of polyhedral products. We begin with some definitions (see, for example, [BP2, Chapter 1]).

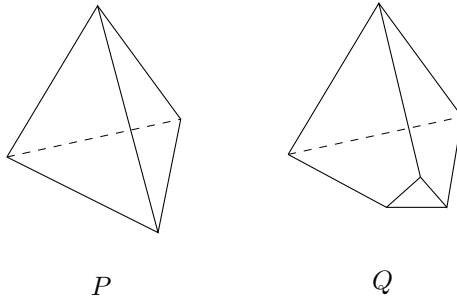
A (*convex*) *polytope* is the convex hull of a finite set of points in \mathbb{R}^n . Its dimension is the dimension of its affine hull. Let P be a d -dimensional polytope. A *facet* of P is a $(d-1)$ -dimensional face. The polytope P is *simple* if each vertex lies in exactly d facets of P . A partial ordering may be defined on the faces of P by inclusion. This determines a poset called the *face poset* of P . The opposite poset, given by reversing the order, determines another polytope P^* called the *dual* of P . If P is simple then P^* is a simplicial complex. Dualizing has the property that $P^{**} = P$. Let ∂P^* be the boundary of P^* .

Suppose that P is a simple polytope. Following [BP2, DJ], a moment-angle complex $\mathcal{Z}(P)$ can be associated to P by defining $\mathcal{Z}(P) = \mathcal{Z}_{\partial P^*}$. Generalizing to polyhedral products in the case where each coordinate space equals a common space X , define $(CX, X)(P)$ as $(CX, X)^{\partial P^*}$. The moment-angle complex $\mathcal{Z}(P)$ is in fact a manifold, but this property does not extend in general to $(CX, X)(P)$.

An operation that produces new simple polytopes from existing ones is by doing vertex cuts.

Definition 3.1. Let P be a simple polytope of dimension d and let $V(P)$ be its vertex set. A hyperplane H in \mathbb{R}^d cuts a vertex x of P if x and $V(P)/\{x\}$ lie in different open half-spaces of H . Let Q be the intersection of P with the closed half-space of H containing $V(P)/\{x\}$. We say that Q is obtained from P by a *vertex cut operation*.

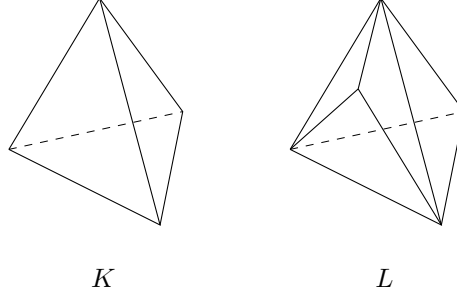
Diagrammatically, this is pictured as follows:



The dual of a vertex cut operation is a stacking operation.

Definition 3.2. Let K be a simplicial complex of dimension d and let σ be a facet of K . Define L as $K \cup_{\sigma} \Delta^d$, that is, L is obtained from K by gluing a d -simplex onto K along the facet σ . We say that L is obtained from K by a *stacking operation*.

Diagrammatically, this is pictured as follows:



Notice that it is immediate from the definitions that the vertex cut and stacking operations preserve dimension.

The objects we wish to study are the moment-angle manifold $\mathcal{Z}(P)$ and the polyhedral product $(CX, X)(P)$ where P is a simple polytope obtained from Δ^d by iterated vertex cut operations. Equivalently, we study the polyhedral products $\mathcal{Z}_{\partial P^*}$ and $(CX, X)^{\partial P^*}$ where ∂P^* is the boundary of a simple polytope obtained from Δ^d by iterated stacking operations.

An important property of the vertex cut operation is that the diffeomorphism type of $\mathcal{Z}(P)$ is independent of the order in which the vertices were cut [GL, Theorem 2.1]. Dually, the diffeomorphism type of $\mathcal{Z}(P)$ is independent of the stacking order for P^* . Weakening to homotopy type, we generalize this property to polyhedral products.

Proposition 3.3. *Let P be a simple polytope and let Q be a simple polytope obtained from P by a vertex cut operation. Then the homotopy type of $(CX, X)(Q)$ is independent of which vertex was cut.*

Remark 3.4. It is easy to see that Proposition 3.3 does not hold when (CX, X) is replaced by $(\underline{CX}, \underline{X})$, that is, when the coordinate spaces X_i may be different. For example, let $P = \Delta^2$ with vertex set $\{1, 2, 3\}$. Cut vertex 1 to obtain a new polytope Q_1 on the vertex set $\{2, 3, 4, 5\}$ or cut vertex 2 to obtain a new polytope Q_2 on the vertex set $\{1, 3, 4, 5\}$. Both Q_1 and Q_2 equal the square I^2 . Notice that Q_1 and Q_2 are self-dual, so $\partial Q_1^* = \partial Q_2^* = \partial I^2$. Next, observe that $\partial I^2 = A * B$ where A and B are 2 points and $*$ is the join operation, defined in general by $K_1 * K_2 = \{\sigma_1 \cup \sigma_2 \mid \sigma_i \in K_i\}$. A straightforward property of the polyhedral product [BBCG] is that $(\underline{CX}, \underline{X})^{K_1 * K_2} \simeq (\underline{CX}, \underline{X})^{K_1} \times (\underline{CX}, \underline{X})^{K_2}$. In our case, this gives $(\underline{CX}, \underline{X})^{\partial I^2} = (\underline{CX}, \underline{X})^A \times (\underline{CX}, \underline{X})^B$. Therefore, taking coordinate spaces X_i for $1 \leq i \leq 5$, we obtain $(\underline{CX}, \underline{X})^{\partial Q_1^*} \simeq (X_2 * X_3) \times (X_4 * X_5)$ while $(\underline{CX}, \underline{X})^{\partial Q_2^*} \simeq (X_1 * X_3) \times (X_4 * X_5)$. These have distinct homotopy types, but if each X_i equals a common space X then $(CX, X)(Q_1) = (CX, X)^{\partial Q_1^*} \simeq (CX, X)^{\partial Q_2^*} = (CX, X)(Q_2)$.

We will prove the equivalent, dual statement to Proposition 3.3.

Proposition 3.5. *Let K be a simplicial complex of dimension d which is dual to a simple polytope P . Let L be a simplicial complex obtained from K by stacking along a facet of K . Then the homotopy type of $(CX, X)^{\partial L}$ is independent of which facet of K was stacked.*

Proof. Let σ_1 and σ_2 be two facets of K . For $t = 1, 2$, let Δ_t^d be a d -simplex stacked onto σ_t . Then there are pushouts

$$\begin{array}{ccc} \sigma_t & \longrightarrow & \Delta_t^d \\ \downarrow & & \downarrow \\ K & \longrightarrow & L_t \end{array}$$

which define the simplicial complexes L_1 and L_2 . Since σ_1 and σ_2 are faces of ∂K , the stacking operation also induces pushouts

$$\begin{array}{ccc} \sigma_t & \longrightarrow & \partial \Delta_t^d \\ \downarrow & & \downarrow \\ \partial K & \longrightarrow & \partial L_t. \end{array}$$

We will prove the proposition by showing that $(CX, X)^{\partial L_1} \simeq (CX, X)^{\partial L_2}$.

It is useful to first consider $(\underline{CX}, \underline{X})^{\partial L_1}$ and $(\underline{CX}, \underline{X})^{\partial L_2}$ where we have to more explicitly keep track of coordinates. Suppose that the vertex set of K is $\{1, \dots, m\}$. Stacking introduces one additional vertex in L_t which we label in both cases as $m+1$. Suppose that σ_t is on the vertex set $\{i_{t,1}, \dots, i_{t,d}\}$. Let $\{j_{t,1}, \dots, j_{t,m-d}\}$ be the complement of $\{i_{t,1}, \dots, i_{t,d}\}$ in $\{1, \dots, m\}$. Observe that Δ_t^d is on the vertex set $\{i_{t,1}, \dots, i_{t,d}, m+1\}$. Regarding each of σ_t , Δ_t^d , K and L_t as being on the vertex set $\{1, \dots, m+1\}$, we can take polyhedral products to obtain pushouts

$$(4) \quad \begin{array}{ccc} (\underline{CX}, \underline{X})^{\overline{\sigma_t}} & \longrightarrow & (\underline{CX}, \underline{X})^{\overline{\partial \Delta_t^d}} \\ \downarrow & & \downarrow \\ (\underline{CX}, \underline{X})^{\overline{\partial K}} & \longrightarrow & (\underline{CX}, \underline{X})^{\partial L_t}. \end{array}$$

In general, if τ is a d -simplex on the vertex set $\{1, \dots, d+1\}$, then by definition of the polyhedral product we have $(\underline{CX}, \underline{X})^\tau = \prod_{r=1}^{d+1} CX_r$, and $(\underline{CX}, \underline{X})^{\partial \tau} = \bigcup_{r=1}^{d+1} (CX_1 \times \dots \times X_r \times \dots \times CX_{d+1})$, where, in each term of the union, all the factors are cones except for one. Applying this to our case, we obtain

$$\begin{aligned} (\underline{CX}, \underline{X})^{\overline{\sigma_t}} &= (\underline{CX}, \underline{X})^{\sigma_t} \times \prod_{s=d+1}^m X_{j_{t,s}} \times X_{m+1} = \prod_{s=1}^d CX_{i_{t,s}} \times \prod_{s=d+1}^m X_{j_{t,s}} \times X_{m+1} \\ (\underline{CX}, \underline{X})^{\overline{\partial \Delta_t^d}} &= (\underline{CX}, \underline{X})^{\partial \Delta_t^d} \times \prod_{s=d+1}^m X_{j_{t,s}} = \bigcup_{s=1}^{d+1} (CX_{i_{t,1}} \times \dots \times X_{i_{t,s}} \times \dots \times CX_{i_{t,d+1}}) \times \prod_{s=d+1}^m X_{j_{t,s}} \\ (\underline{CX}, \underline{X})^{\overline{K}} &= (\underline{CX}, \underline{X})^K \times X_{m+1} \end{aligned}$$

where, in the second line, to compress notation we have used $i_{t,d+1}$ to refer to the vertex $m+1$ for both $t=1,2$. Further, under these identifications, the map $(\underline{CX}, \underline{X})^{\overline{\sigma_t}} \rightarrow (\underline{CX}, \underline{X})^{\overline{\partial K}}$ is the product of the identity map on X_{m+1} and a map $f_t: (\underline{CX}, \underline{X})^{\sigma_t} \times \prod_{s=d+1}^m X_{j_{t,s}} \rightarrow (\underline{CX}, \underline{X})^{\partial K}$ induced by the inclusion of the face $\sigma_t \rightarrow \partial K$, and the map $(\underline{CX}, \underline{X})^{\overline{\sigma_t}} \rightarrow (\underline{CX}, \underline{X})^{\overline{\partial \Delta^d}}$ is a coordinate-wise inclusion which we label as i . Thus (4) can be identified with the pushouts

$$(5) \quad \begin{array}{ccc} \left(\prod_{s=1}^d CX_{i_{t,s}} \times \prod_{s=d+1}^m X_{j_{t,s}} \right) \times X_{m+1} & \xrightarrow{i} & \bigcup_{s=1}^{d+1} (CX_{i_{t,1}} \times \cdots \times X_{i_{t,s}} \times \cdots \times CX_{i_{t,d+1}}) \\ \downarrow f_t \times 1 & & \downarrow \\ (\underline{CX}, \underline{X})^{\partial K} \times X_{m+1} & \xrightarrow{\quad\quad\quad} & (\underline{CX}, \underline{X})^{\partial L_t}. \end{array}$$

Now simplifying to the case of (CX, X) when each coordinate space X_i equals a common space X , we obtain pushouts

$$(6) \quad \begin{array}{ccc} \left(\prod_{s=1}^d CX \times \prod_{s=d+1}^m X \right) \times X & \xrightarrow{i} & \bigcup_{s=1}^{d+1} (CX \times \cdots \times X \times \cdots \times CX) \\ \downarrow f_t \times 1 & & \downarrow \\ (CX, X)^{\partial K} \times X & \xrightarrow{\quad\quad\quad} & (CX, X)^{\partial L_t}. \end{array}$$

Observe that the only difference in the pushouts for $(CX, X)^{\partial L_1}$ and $(CX, X)^{\partial L_2}$ in (6) are the maps f_1 and f_2 . We will show that there is a self-homotopy equivalence e of $(CX, X)^{\partial K}$ which satisfies a homotopy commutative square

$$(7) \quad \begin{array}{ccc} \prod_{s=1}^d CX \times \prod_{s=d+1}^m X & \xrightarrow{p} & \prod_{s=1}^d CX \times \prod_{s=d+1}^m X \\ \downarrow f_1 & & \downarrow f_2 \\ (CX, X)^{\partial K} & \xrightarrow{e} & (CX, X)^{\partial K} \end{array}$$

where p permutes coordinates. Granting this, observe that we obtain a map from the $t=1$ pushout in (6) to the $t=2$ pushout by using $p \times 1$ on the upper left corner, $e \times 1$ on the lower right corner, and noting that i is a coordinate-wise inclusion, we can also use $p \times 1$ on the upper right corner. This induces a map of pushouts $h: (CX, X)^{\partial L_1} \rightarrow (CX, X)^{\partial L_2}$. As p and e are homotopy equivalences, so is h , and this completes the proof.

It remains to construct the self-homotopy equivalence e of $(CX, X)^{\partial K}$. First consider the simple polytope P that is dual to K . Let v_1 and v_2 be vertices of P . Consider the permutation that interchanges v_1 and v_2 while leaving the other vertices fixed. Since the polytope P is simple, this permutation induces a self-map of the face poset of P which permutes the k -dimensional faces for each $0 \leq k \leq d$. Dually, the face poset for K is obtained by reversing the arrows on the face poset for P , so we obtain a self-map of the face poset of K which permutes the k -dimensional faces for each $0 \leq k \leq d$. Consequently, if we let v_1 and v_2 be the vertices of P that are dual to the facets σ_1 and σ_2 of K , we obtain a map $g': K \rightarrow K$ of simplicial complexes which permutes the facets σ_1 and σ_2 .

This induces a map $g: \partial K \rightarrow \partial K$ of simplicial complexes which permutes the faces σ_1 and σ_2 . Now apply the polyhedral product (CX, X) to the face poset of K . Any face τ of K has $(CX, X)^\tau$ equal to a product of copies of CX or X , depending on whether a vertex is in or not in τ . So the self-map of the face poset of K induces a self-map of $(\underline{CX}, \underline{X})^\tau$ for each face τ of K which permutes the CX factors and permutes the X factors. Any such permutation is a homotopy equivalence. The morphism of face posets ensures that these permutations are compatible under face-wise inclusions, so there are induced maps $e': (CX, X)^K \rightarrow (CX, X)^K$ and $e: (CX, X)^{\partial K} \rightarrow (CX, X)^{\partial K}$ which are homotopy equivalences, and e satisfies (7). \square

Starting with a simplicial complex K of dimension d , there are many ways of iteratively stacking to produce a new simplicial complex L . A particular sequence of stacks is called a *stack history* of L .

Corollary 3.6. *Let K be a simplicial complex of dimension d which is dual to a simple polytope P and let L be a simplicial complex obtained from K by iterated stacking operations. Then the homotopy type of $(CX, X)^{\partial L}$ is independent of the stack history of L .* \square

4. DELETING A VERTEX FROM THE BOUNDARY OF A STACKED POLYTOPE

In this section we consider a special case of iterated stacking operations. Let $P = \Delta^d$ be the d -simplex. Then P is a simple polytope, and the dual $K = P^*$ of P is again Δ^d . In this case, if L is obtained from K by a sequence of stacking operations, then L is also a simple polytope of dimension d , as well as a simplicial complex. The simple polytope L is called a *stacked polytope*. Each copy of Δ^d in L is called a *stack*, so if L is formed by $\ell - 1$ stacking operations, then it has ℓ stacks.

Suppose that L is a stacked polytope with ℓ stacks. So there is a sequence of stacked polytopes

$$L_1 = \Delta^d \subseteq L_2 \subseteq \cdots \subseteq L_\ell = L$$

where, for $2 \leq i \leq \ell$, L_i has been formed by gluing a Δ^d to L_{i-1} along a common facet. By Corollary 3.6, the homotopy type of $(CX, X)^{\partial L}$ is independent of the stack history of L . Thus we can choose a stacking order which is more convenient for analyzing $(CX, X)^{\partial L}$.

The prescribed stacking order we choose is as follows. Let $\mathcal{L}_1 = \Delta^d$. Label the vertices of Δ^d as $\{1, \dots, d+1\}$. Form \mathcal{L}_2 by stacking a copy of Δ^d to \mathcal{L}_1 on the facet $(1, \dots, d)$. Label the one extra vertex of \mathcal{L}_2 as $d+2$, and notice that if $d = 1$ then the vertex $\{3\}$ is a facet of \mathcal{L}_2 and if $d > 1$ then $(1, \dots, d-1, d+2)$ is a facet of \mathcal{L}_2 . Now stack onto this facet and iterate the procedure. We obtain, for $2 < k \leq \ell$, a stacked polytope \mathcal{L}_{k-1} on the vertex set $\{1, \dots, d+k-1\}$ where if $d = 1$ then the vertex $\{k = d+k-1\}$ is a facet and if $d > 1$ then $(1, \dots, d-1, d+k-1)$ is a facet. Form \mathcal{L}_k by stacking a copy of Δ^d on this facet. Label the one extra vertex of \mathcal{L}_k as $d+k$, and observe that if $d = 1$ then the vertex $\{d+k\}$ is a facet of \mathcal{L}_k and if $d > 1$ then $(1, \dots, d-1, d+k)$ is a facet of \mathcal{L}_k . Finally, let $\mathcal{L} = \mathcal{L}_\ell$.

Now we identify the simplicial complex obtained by deleting the vertex $\{1\}$ from $\partial\mathcal{L}$.

Lemma 4.1. *The simplicial complex $\mathcal{L} = \mathcal{L}_\ell$ has the following properties:*

- (a) \mathcal{L} has $\ell d + 1$ facets;
- (b) there are ℓ facets in part (a) which do not contain the vertex $\{1\}$: these are $(2, 3, \dots, d, d+1)$, $(2, 3, \dots, d, d+2)$, and for $2 < k \leq \ell$, $(2, 3, \dots, d-1, d+k-1, d+k)$;
- (c) the simplicial complex $\partial\mathcal{L} - \{1\}$ filters as a sequence

$$M_1 = \Delta^{d-1} \subseteq M_2 \subseteq \dots \subseteq M_\ell = \partial\mathcal{L} - \{1\}$$

where, for $2 \leq k \leq \ell$, $M_k = M_{k-1} \cup_{\sigma_k} \Delta^{d-1}$, where $\sigma_k = \Delta^{d-2}$.

Proof. For part (a), observe that $\mathcal{L}_1 = \Delta^d$ has $d+1$ facets. As \mathcal{L}_ℓ is formed by gluing on $(\ell-1)$ more Δ^d 's, the total of ℓ copies of Δ^d have $\ell(d+1)$ facets. But each gluing occurs along a common facet, so at each of the $(\ell-1)$ gluings 1 facet is removed. Thus \mathcal{L}_ℓ has $\ell(d+1) - (\ell-1) = \ell d + 1$ facets.

For part (b), observe that in $\mathcal{L}_1 = \Delta^d$ there are $d+1$ facets but only one of them, $(2, 3, \dots, d, d+1)$, does not contain the vertex $\{1\}$. In forming \mathcal{L}_2 , we stack on the facet $(1, 2, \dots, d)$ of \mathcal{L}_1 , and label the extra vertex $d+2$. This operation removes $(1, 2, \dots, d)$ as a facet of \mathcal{L}_1 and introduces d new facets: all $d+1$ facets of Δ^d on the vertex set $\{1, 2, \dots, d, d+2\}$ except for $(1, 2, \dots, d)$. Of the new facets, only one of them, $(2, 3, \dots, d, d+2)$, does not contain the vertex $\{1\}$. Iterating, for $2 < k \leq \ell$, in forming \mathcal{L}_k , we stack on the facet $(1, 2, \dots, d-1, d+k-1)$ of \mathcal{L}_{k-1} , and label the extra vertex $d+k$. This operation removes $(1, 2, \dots, d-1, d+k-1)$ as a facet of \mathcal{L}_k and introduces d new facets: all $d+1$ facets of Δ^d on the vertex set $\{1, 2, \dots, d-1, d+k-1, d+k\}$ except for $(1, 2, \dots, d-1, d+k-1)$. Of the new facets, only one of them, $(2, 3, \dots, d-1, d+k-1, d+k)$, does not contain the vertex $\{1\}$. Thus, precisely ℓ of the $\ell d + 1$ total facets of \mathcal{L}_ℓ do not contain the vertex $\{1\}$, and these are: $(2, 3, \dots, d, d+1)$, $(2, 3, \dots, d, d+2)$, and for $2 < k \leq \ell$, $(2, 3, \dots, d-1, d+k-1, d+k)$.

For part (c), since \mathcal{L} is a simple polytope which is also a simplicial complex, the geometric realization of $\partial\mathcal{L}$ can be obtained by gluing together the facets of \mathcal{L} . The geometric realization of the simplicial complex $\partial\mathcal{L} - \{1\}$ is therefore obtained by gluing together those facets of \mathcal{L} which do not contain the vertex $\{1\}$. We perform this gluing procedure one simplex at a time. Let $M_1 = \Delta^{d-1}$ be $(2, 3, \dots, d, d+1)$. Form M_2 by gluing the $(d-1)$ -simplex $(2, 3, \dots, d, d+2)$ to M_1 along the common $(d-2)$ -simplex $(2, 3, \dots, d)$. For $2 \leq k \leq \ell$, form M_k by gluing the $(d-1)$ -simplex $(2, 3, \dots, d-1, d+k-1, d+k)$ to M_{k-1} along the common $(d-2)$ -simplex $(2, 3, \dots, d-1, d+k-1)$. Then $M_\ell = \partial\mathcal{L} - \{1\}$. \square

Applying Proposition 2.4 to Lemma 4.1 (c), we immediately obtain the following.

Proposition 4.2. *There is a homotopy equivalence*

$$(CX, X)^{\partial\mathcal{L}-\{1\}} \simeq (CX, X)^{P_\ell}$$

where P_ℓ is ℓ disjoint points. \square

Now specialize to polyhedral products on the pairs (D^2, S^1) and write \mathcal{Z}_K for $(CX, X)^K$. In [GT1] the homotopy type of \mathcal{Z}_{P_ℓ} was identified, giving the following.

Corollary 4.3. *There is a homotopy equivalence*

$$\mathcal{Z}_{\partial\mathcal{L}-\{1\}} \simeq \mathcal{Z}_{P_\ell} \simeq \bigvee_{k=3}^{\ell+1} (S^k)^{\wedge^{(k-2)}\binom{\ell}{k-1}}.$$

\square

5. CUP PRODUCTS IN $H^*(\mathbb{Z}_{\partial\mathcal{L}})$

On the one hand, since \mathcal{L} is a stacked polytope of dimension d with ℓ stacks, it is dual to a simple polytope obtained from Δ^d by $\ell - 1$ vertex cuts. So by [BM, GL] there is a diffeomorphism $\mathcal{Z}_{\partial\mathcal{L}} \cong \#_{k=3}^{\ell+1} (S^k \times S^{\ell+2d-k})^{\#^{(k-2)}\binom{\ell}{k-1}}$. The cup products in $H^*(\mathcal{Z}_{\partial\mathcal{L}})$ are then clear from the description of the space as a connected sum of products of spheres. On the other hand, there is a combinatorial description of the cup product structure in \mathcal{Z}_K for any simplicial complex K , proved in [BBP, BP1, F]. Take homology with integer coefficients. The *join* of two simplicial complexes K_1 and K_2 is $K_1 * K_2 = \{\sigma_1 \cup \sigma_2 \mid \sigma_i \in K_i\}$.

Theorem 5.1. *There is an isomorphism of graded commutative algebras*

$$H^*(\mathcal{Z}_K) \cong \bigoplus_{I \subset [m]} \tilde{H}^*(K_I).$$

Here, $\tilde{H}^*(K_I)$ denotes the reduced simplicial cohomology of the full subcomplex $K_I \subset K$ (the restriction of K to $I \subset [m]$). The isomorphism is the sum of isomorphisms

$$H^p(\mathcal{Z}_K) \cong \sum_{I \subset [m]} \tilde{H}^{p-|I|-1}(K_I)$$

and the ring structure (the Hochster ring) is given by the maps

$$H^{p-|I|-1}(K_I) \otimes H^{q-|J|-1}(K_J) \longrightarrow H^{p+q-|I|-|J|-1}(K_{I \cup J})$$

which are induced by the canonical simplicial maps $K_{I \cup J} \longrightarrow K_I * K_J$ for $I \cap J = \emptyset$ and zero otherwise. \square

Theorem 5.1 implies that the Hochster ring structure on $\mathcal{Z}_{\partial\mathcal{L}}$ matches the ring product structure arising from the geometry of the connected sum, at least up to an isomorphism. We need information from both, so we are led to geometrically realize the isomorphism, via a homotopy equivalence.

In general, if M is an n -dimensional manifold, let $M - *$ be M with a point in the interior of the n -disc removed. As a CW -complex, $M - *$ is homotopy equivalent to the $(n - 1)$ -skeleton of M . By definition of the connected sum, if M and N are two n -dimensional manifolds then

$(M \# N) - * \simeq (M - *) \vee (N - *)$. In our case, as $\mathcal{Z}_{\partial \mathcal{L}} = \#_{k=3}^{\ell+1}(S^k \times S^{\ell+2d-k})^{\#(k-2)} \binom{\ell}{k-1}$, the $(\ell+2d-1)$ -skeleton of $\mathcal{Z}_{\partial \mathcal{L}}$ is the wedge

$$W = \bigvee_{k=3}^{\ell+1} \left(\bigvee_{t=1}^{(k-1) \binom{\ell}{k-1}} (S^k \vee S^{\ell+2d-k}) \right).$$

Therefore, there is one ring generator in $H^*(\mathcal{Z}_{\partial \mathcal{L}})$ for each sphere in the wedge W .

Applying Theorem 5.1 to $\mathcal{Z}_{\partial \mathcal{L}}$ we obtain an abstract isomorphism of algebras $h: H^*(\mathcal{Z}_{\partial \mathcal{L}}) \rightarrow H^*(\mathcal{Z}_{\partial \mathcal{L}})$, where on the left the generating set is given by the Hochster ring structure, on the right the generating set is given by the CW -structure of $\mathcal{Z}_{\partial \mathcal{L}}$, and h maps generators to generators. Restricting to degrees less than $\ell+2d$, we obtain an abstract isomorphism of modules $h': H^*(W) \rightarrow H^*(W)$. Dualizing, we obtain an abstract isomorphism of modules $h'': H_*(W) \rightarrow H_*(W)$. Since W is a wedge of spheres, the abstract map h'' may be realized geometrically, as follows. Let n be the number of spheres in the wedge W and label the spheres from $1, \dots, n$. For $1 \leq i \leq n$, let $j_i: S_n \rightarrow W$ be the inclusion into the wedge, and let $x_i \in H_*(W)$ be the Hurewicz image of j_i . Suppose that $h''(x_i) = t_{i,1}x_1 + \dots + t_{i,n}x_n$ for some integers $t_{i,1}, \dots, t_{i,n}$. Define $g_i: S_i \rightarrow W$ by $g_i = t_{i,1}j_1 + \dots + t_{i,n}j_n$. Let $g: W \rightarrow W$ be the wedge sum of the maps g_i for $1 \leq i \leq n$. Then $g_* = h''$. Dualizing, $g^* = h'$. As h' is an isomorphism, by Whitehead's Theorem g is a homotopy equivalence.

Next, the map attaching the top cell to W to form $\mathcal{Z}_{\partial \mathcal{L}}$ is a sum of Whitehead products, one Whitehead product for each $S^k \times S^{\ell+2d-k}$. This Whitehead product is detected in cohomology by a nonzero cup product. Since Theorem 5.1 gives a ring isomorphism between the cup product structures on $H^*(\mathcal{Z}_{\partial \mathcal{L}})$ from the connected sum and the Hochster ring, g can be extended to a map

$$\Gamma: \mathcal{Z}_{\partial \mathcal{L}} \rightarrow \mathcal{Z}_{\partial \mathcal{L}}$$

which induces an isomorphism in cohomology and so is a homotopy equivalence. Thus we have the following.

Lemma 5.2. *Altering $\mathcal{Z}_{\partial \mathcal{L}} \cong \#_{k=3}^{\ell+1}(S^k \times S^{\ell+2d-k})^{\#(k-2)} \binom{\ell}{k-1}$ by a self-homotopy equivalence if necessary, we may assume that each Hochster ring generator in $H^*(\mathcal{Z}_{\partial \mathcal{L}})$ is represented by a map $S^t \rightarrow \#_{k=3}^{\ell+1}(S^k \times S^{\ell+2d-k})^{\#(k-2)} \binom{\ell}{k-1}$ which is the inclusion of one of the spheres in the $(\ell+2d-1)$ -skeleton of the connected sum.* \square

Lemma 5.2 lets us use combinatorial information from the Hochster ring to deduce cup product information for the cohomology of the connected sum. We apply this to deduce some cup product information in $H^*(\mathcal{Z}_{\partial \mathcal{L}})$.

Let \mathcal{I} be an index set which runs over all the products of two spheres in the connected sum $\#_{k=3}^{\ell+1}(S^k \times S^{\ell+2d-k})^{\#(k-2)} \binom{\ell}{k-1}$. There are $\sum_{k=3}^{\ell+1} (k-1) \binom{\ell}{k-1}$ elements in \mathcal{I} . Each $\alpha \in \mathcal{I}$ corresponds to a product of spheres $S^k \times S^{\ell+2d-k}$ which determines a nontrivial cup product in $H^*(\mathcal{Z}_{\partial \mathcal{L}})$: if $x_\alpha, y_\alpha \in H^*(\mathcal{Z}_{\partial \mathcal{L}})$ are generators corresponding to the inclusions of S^k and $S^{\ell+2d-k}$ into the

$(\ell + 2d - 1)$ -skeleton of the connected sum, then $x_\alpha \cup y_\alpha \neq 0$. By Lemma 5.2, we may assume that x_α and y_α are Hochster ring generators. Thus $x_\alpha \in \tilde{H}^*(\partial\mathcal{L}_{I_\alpha})$ and $y_\alpha \in \tilde{H}^*(\partial\mathcal{L}_{J_\alpha})$ for some index sets I_α and J_α of $[m]$, where $m = \ell + d$ is the number of vertices of $\partial\mathcal{L}$.

The inclusion $\partial\mathcal{L} - \{1\} \rightarrow \partial\mathcal{L}$ induces a map

$$f: \mathcal{Z}_{\partial\mathcal{L}-\{1\}} \rightarrow \mathcal{Z}_{\partial\mathcal{L}}.$$

Lemma 5.3. *Ker f^* contains one and only one of x_α or y_α .*

Proof. First, in the Hochster ring for $H^*(\mathcal{Z}_{\partial\mathcal{L}})$ we have $x_\alpha \cup y_\alpha \neq 0$. So by Theorem 5.1, $I_\alpha \cap J_\alpha = \emptyset$.

We claim that $1 \in I_\alpha \cup J_\alpha$. For if not, then $I_\alpha \cup J_\alpha$ is contained in the vertex set for $\partial\mathcal{L} - \{1\}$. Therefore, by Theorem 5.1 all three of $\tilde{H}^*(\partial\mathcal{L}_{I_\alpha})$, $\tilde{H}^*(\partial\mathcal{L}_{J_\alpha})$, $\tilde{H}^*(\partial\mathcal{L}_{I_\alpha \cup J_\alpha})$ are contained in $H^*(\mathcal{Z}_{\partial\mathcal{L}-\{1\}})$. That is, $x_\alpha, y_\alpha \in H^*(\mathcal{Z}_{\partial\mathcal{L}-\{1\}})$ and so $x_\alpha \cup y_\alpha \in H^*(\mathcal{Z}_{\partial\mathcal{L}-\{1\}})$. But by Corollary 4.3, $\mathcal{Z}_{\partial\mathcal{L}-\{1\}}$ is homotopy equivalent to a wedge of spheres, implying that all the cup products in its cohomology are zero, a contradiction.

Now, since $1 \in I_\alpha \cup J_\alpha$ and $I_\alpha \cap J_\alpha = \emptyset$, either $1 \in I_\alpha$ or $1 \in J_\alpha$. If $1 \in I_\alpha$ then $1 \notin J_\alpha$, implying that $x_\alpha \in \tilde{H}^*(\partial\mathcal{L}_I)$ is not an element of $H^*(\mathcal{Z}_{\partial\mathcal{L}-\{1\}})$ while $y_\alpha \in \tilde{H}^*(\partial\mathcal{L}_J)$ is. That is, $f^*(x_\alpha) = 0$ while $f^*(y_\alpha) \neq 0$. Similarly, if $1 \in J_\alpha$ then $f^*(x_\alpha) \neq 0$ and $f^*(y_\alpha) = 0$. \square

6. PANOV'S PROBLEM

Recall from the Introduction that if K is ℓ disjoint points then there is a homotopy equivalence

$$(8) \quad \mathcal{Z}_K \simeq \bigvee_{k=3}^{\ell+1} (S^k)^{\wedge(k-2)} \binom{\ell}{k-1}$$

and if P is a simple polytope of dimension d obtained from Δ^d by $\ell - 1$ vertex cut operations (in any order) then there is a diffeomorphism

$$(9) \quad \mathcal{Z}(P) \cong \#_{k=3}^{\ell+1} (S^k \times S^{\ell+2d-k})^{\#(k-2)} \binom{\ell}{k-1}.$$

Panov posed the problem of identifying the nature of the correspondence between the decompositions in (8) and (9). In this section we give an answer to the problem.

Let P be a simple polytope of dimension d which has been obtained from Δ^d by $\ell - 1$ vertex cuts. Dualizing, P^* is a stacked polytope of dimension d with ℓ stacks. By Proposition 3.5, the homotopy type of $\mathcal{Z}(P) = \mathcal{Z}_{\partial P^*}$ is independent of the stacking order of P^* . We may therefore analyze the homotopy type of $\mathcal{Z}(P)$ by analyzing the homotopy type of $\mathcal{Z}_{\partial\mathcal{L}}$.

Proof of Theorem 1.1. Consider the inclusion

$$\partial\mathcal{L} - \{1\} \rightarrow \partial\mathcal{L}.$$

The moment-angle complex, regarded as a polyhedral product, is natural for maps of simplicial complexes, so we obtain an induced map of moment-angle complexes

$$f: \mathcal{Z}_{\partial\mathcal{L}-\{1\}} \longrightarrow \mathcal{Z}_{\partial\mathcal{L}}.$$

By Corollary 4.3 and (9), up to homotopy equivalences f can be regarded as a map

$$f: \bigvee_{k=3}^{\ell+1} (S^k)^{\wedge(k-2)\binom{\ell}{k-1}} \longrightarrow \#_{k=3}^{\ell+1} (S^k \times S^{\ell+2d-k})^{\#(k-2)\binom{\ell}{k-1}}.$$

In general, whenever K' is a full subcomplex of K , by [BBCG] there is a retract of $(\underline{CX}, \underline{X})^{K'}$ off of $(\underline{CX}, \underline{X})^K$. In our case, since $\partial\mathcal{L} - \{1\}$ is a full subcomplex of $\partial\mathcal{L}$, the map f has a left homotopy inverse

$$g: \#_{k=3}^{\ell+1} (S^k \times S^{\ell+2d-k})^{\#(k-2)\binom{\ell}{k-1}} \longrightarrow \bigvee_{k=3}^{\ell+1} (S^k)^{\wedge(k-2)\binom{\ell}{k-1}}.$$

This proves parts (a), (b) and (c) of the theorem. Part (d) is Lemma 5.3. \square

More is true than stated in Theorem 1.1, and it may be useful to elaborate on it. As in Section 5, let \mathcal{I} be an index set which runs over all the products of two spheres in the connected sum (9). Each $\alpha \in \mathcal{I}$ corresponds to a product of spheres $S^k \times S^{\ell+2d-k}$ which determines a nontrivial cup product in $H^*(\mathcal{Z}_{\partial\mathcal{L}})$: if $x_\alpha, y_\alpha \in H^*(\mathcal{Z}_{\partial\mathcal{L}})$ are generators corresponding to the spheres $S^k \vee S^{\ell+2d-k} \subset S^k \times S^{\ell+2d-k}$ then $x_\alpha \cup y_\alpha \neq 0$. By Proposition 5.3, f^* is nonzero for one and only one of x_α or y_α . It is not immediately clear which of x_α or y_α is sent nontrivially by f_* to $H^*(\mathcal{Z}_{\partial\mathcal{L}-\{1\}})$, so write z_α for the generator which has nontrivial image. By Lemma 5.2, z_α is the dual of the Hurewicz image of the composite of inclusions

$$i_\alpha: S^{t_\alpha} \hookrightarrow S^k \vee S^{\ell+2d-k} \hookrightarrow \#_{k=3}^{\ell+1} (S^k \times S^{\ell+2d-k})^{\#(k-2)\binom{\ell}{k-1}}$$

where t_α is k or $\ell + 2d - k$ depending on whether z_α is x_α or y_α . Taking the wedge sum of all the maps i_α for every $\alpha \in \mathcal{I}$ we obtain a map

$$i: \bigvee_{\alpha \in \mathcal{I}} S^{t_\alpha} \longrightarrow \#_{k=3}^{\ell+1} (S^k \times S^{\ell+2d-k})^{\#(k-2)\binom{\ell}{k-1}}$$

with the property that i^* factors through f^* . That is, there is a commutative diagram

$$(10) \quad \begin{array}{ccc} & & H^*(\bigvee_{\alpha \in \mathcal{I}} S^{t_\alpha}) \\ & \nearrow \phi & \uparrow i^* \\ H^*(\bigvee_{k=3}^{\ell+1} (S^k)^{\wedge(k-2)\binom{\ell}{k-1}}) & \xleftarrow{f^*} & H^*(\#_{k=3}^{\ell+1} (S^k \times S^{\ell+2d-k})^{\#(k-2)\binom{\ell}{k-1}}) \end{array}$$

for some ring map ϕ . Note at this point that ϕ need not be induced by a map of spaces, it exists only on the level of cohomology.

By construction, i is the inclusion of one factor in each product of spheres in the connected sum. So i^* is an epimorphism taking ring generators to ring generators. The commutativity of (10) therefore implies that ϕ is also an epimorphism, and must take ring generators to ring generators.

Now observe that both $\bigvee_{\alpha \in \mathcal{I}} S^{t_\alpha}$ and $\bigvee_{k=3}^{\ell+1} (S^k)^{\wedge(k-2)} \binom{\ell}{k-1}$ are wedges of precisely the same number of spheres. So the domain and range of ϕ have the same number of ring generators. Hence ϕ must be an isomorphism.

Finally, we geometrically realize ϕ . Consider the composite

$$\bigvee_{\alpha \in \mathcal{I}} S^{t_\alpha} \xrightarrow{i} \#_{k=3}^{\ell+1} (S^k \times S^{\ell+2d-k})^{\#(k-2)} \binom{\ell}{k-1} \xrightarrow{g} \bigvee_{k=3}^{\ell+1} (S^k)^{\wedge(k-2)} \binom{\ell}{k-1}.$$

Taking cohomology, by (10) we obtain $i^* \circ g^* = \phi \circ f^* \circ g_*$. Since g is a left homotopy inverse of f , we therefore have $i^* \circ g^* = \phi$. Since ϕ is an isomorphism, so is $i^* \circ g^*$, implying by Whitehead's Theorem that $g \circ i$ is a homotopy equivalence. Thus ϕ is the map induced in cohomology by the homotopy equivalence $g \circ i$. Note, however, that it may not be the case that there is a homotopy $i \simeq f \circ (g \circ i)$, that is, it may not be the case that (10) can be improved to a homotopy commutative diagram on the level of spaces.

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